❏   135

Exact solution of unsteady MHD free convective flow with constant heat flux: revisited

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Article Info

ABSTRACT

A solution of unstable MHD free convective flow of an incompressible, viscous and electrically steering fluid created by spontaneous motion of vertical plate subjected to constant heat fluctuation offered by Sacheti *et al.* has been revisited. The governing equations are deciphered with the aid of Laplace transform procedure, while the inversion is gotten through the Riemann sum approximation approach. Fluid velocity are varied with sundry parameters such as, Prandtl number, Grashof number and Hartmann number have been extensively explicated with the help of graphs. Numerical comparison is carried out and equated with the benchmark values reported in literature and an outstanding agreement is established. This article targeted at amending some discrepancies in the skin-friction offered by Sacheti *et al*. The benefit of the suggested technique could be a reduction of computation time.

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NOMENCLATURE
B_0 : uniform magnetic field" (Tesla)
C_p : specific heat of the fluid at constant pressure (kJ/kgK)
g : acceleration due to gravity (m/s^2)
G : Grashof number (dimensionless)
m_0 : Hartmann number (dimensionless)
Pr : Prandtl number (dimensionless)
s : Laplace parameter
t : time (secs)
u' : dimensional velocity of the fluid (m/s)
$\boldsymbol{u_0}$ **:** dimensionless velocity of the fluid (dimensionless)

Greek letters
σ : electrical conductive ($Ohms/mK$)
κ : thermal conductivity (W/mK)"
T' : dimensional fluid temperature (K)"
β : volumetric coefficient of thermal expansion ($1/K$)
q : constant heat flux per unit area at the plate (W/m^2)
v : kinematic viscosity (m^2/s)
τ : skin friction (dimensionless)
ρ : density (kg/m^3)

1. INTRODUCTION

The study of hydromagnetic convective flow has drawn serious attention over the decades based on its scientific and technological applications, such as; nuclear reactors, spacecraft design, geothermal energy extraction, boundary flow layer control, plasma studies, metallurgy, mineral, petroleum engineering, chemical and solar energy collectors. Numerous studies have been conveyed on magnetohydrodynamics flow with diverse physical conditions. Soundalgekar and Patil [1] gave analytical study on stokes problem for infinite vertical plate with constant heat flux. The unsteady hydromagnetic free convection flow with constant heat flux was considered by Sacheti *et al.* [2]. In their work, by using the method of convolution integration, they concluded that the magnetic field has a impeding effect on the velocity. Kafousias *et al.* [3] extended their finding by incorporating the role of the free-convection flow past an impulsively vertical porous plate with constant suction. Ibrahim *et al.* [4] investigated the influence "of chemical reaction and radiation absorption on the unsteady MHD free convection flow past a semi-infinite vertical permeable moving plate with heat source and suction". Also, Anand *et al.* [5] discussed the influence of radiation on an unsteady MHD free convective flow past a vertical porous plate in the presence of soret. Asma *et al.* [6] examined the unsteady MHD free convection flow of cassion fluid past over an oscillating vertical plate embedded in a porous medium while Israel *et al.* [7], further extended the work of [6] by investigating the role of "viscous dissipation and rotation on unsteady MHD free convection flow past an infinite heated vertical plate in a porous medium with time dependent suction". In another related work, Singh [8, 9] investigated the "MHD free-convection flow past an accelerated vertical porous plate in a rotating fluid and unsteady free convection flow through a rotating porous medium". The problem for a viscous incompressible and electrically conducting fluid in the presence of a transverse magnetic field was discussed by Nanousis *et al.* [10].

"Some recent works related to the present investigation are found in the literature [11-16]. Jha *et al.* [11] analytically studied the fully developed steady natural convection flow of conducting fluid in a vertical parallel plate microchannel in the presence of transverse magnetic field. The effect of Hartmann number was reported to decrease the volume flow rate. The combined influence of externally applied transverse magnetic field and suction/injection on steady natural convection flow of conducting fluid in a vertical microchannel was carried out by Jha *et al.* [12]. In another work, Jha *et al.* [13] examined the effect of wall surface curvature on transient MHD free convective flow in vertical micro-concentric-annuli. Jha *et al.* [14] studied exact solution of steady fully developed natural convection flow of viscous, incompressible, and electrically conducting fluid in a vertical annular microchannel. Recently, Jha and Aina [15] presented the magnetohydrodynamic (MHD) natural convection flow in a vertical micro-porous-annulus (MPA) in the presence of radial magnetic field. Also, the MHD natural convection flow in vertical micro-concentric-annuli (MCA) in the presence of radial magnetic field has been analyzed by Jha *et al.* [16]".

The objective of this article is to revisit the work of [2] by presenting a semi-analytical solution to the work. The numerical values displays that the present solution calculates the same fluid velocity as presented [2]. In addition, the physical justification and table presented for skin-friction by [2] is wrongly presented, and this makes their analytical result for skin-friction unreliable and incorrect. In view of this, we set out to give a correct expression for skin-friction and justification of result.

2. MATHEMATICAL ANALYSIS

Reexamine the mathematical equations offered by [2]. Following their equations and rules, the governing momentum and energy balance equations describing the current investigation in non-dimensionless forms are [2]:

$$\frac{\partial u'}{\partial t'} = \nu \frac{\partial^2 u'}{\partial y'^2} + g\beta(T' - T'_\infty) - \frac{\sigma B_0^2 u'}{\rho} \tag{1}$$

$$\frac{\partial T'}{\partial t'} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T'}{\partial y'^2} \tag{2}$$

Subjected to the corresponding conditions:

$$t' \leq 0, u' = 0, T' = T'_\infty \text{ for } y' \geq 0$$

$$t' > 0 \begin{cases} u' = u_0, \frac{\partial T'}{\partial y'} = -\frac{q}{\kappa}, & at \ y' = 0 \\ u' \to 0, T' \to T'_\infty, & as \ y' \to \infty \end{cases} \tag{3}$$

Presenting the ensuing dimensionless quantities as shown in (1)-(3)

$$
\left.
\begin{aligned}
&y = \frac{u_0 y'}{v}, u = \frac{u'}{u_0}, t = \frac{u_0{}^2 t'}{v}, T = k u_0 \frac{(T' - T'_\infty)}{(qv)} \\
&\Pr = \frac{v\rho\, C_p}{k}, G = \frac{qg B_0 v^2}{(k u_0{}^4)}, m_0 = \frac{\sigma B_0{}^2 v}{(\rho u_0{}^2)}
\end{aligned}
\right\}
\tag{4}
$$

By (4), (1), and (2) are obtained in dimensionless form as [2]:

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} - m_0 u + GT
\tag{5}
$$

$$
\frac{\partial T}{\partial t} = \Pr \frac{\partial^2 T}{\partial y^2}
\tag{6}
$$

Subject to

$$
t \le 0, u = 0, T = 0, \text{ for } y \ge 0
$$

$$
t > 0
\begin{cases}
u = 1, \dfrac{\partial T}{\partial y} = -1, \ at \ y = 0 \\[2mm]
u' \to 0, T \to 0, \ as \ y \to \infty
\end{cases}
\tag{7}
$$

To decipher (5) with boundary conditions (7), the temperature profile is first obtained from (6) and constituted into (5). By the Laplace transform (LT), the exact solutions are acquired via the LT as follows:

$$
\begin{aligned}
&\bar{u}(y,s) = \int_0^\infty u(y,t)e^{-st}\,dt \\
&\bar{T}(y,s) = \int_0^\infty T(y,t)e^{-st}\,dt
\end{aligned}
\tag{8}
$$

where $s > 0$ and taking the LT of (5) and (6) with the initial and boundary conditions (7), as shown in (5) and (6) become:

$$
\frac{d^2\bar{u}}{dy^2} - (m+s)\bar{u} + G\bar{T} = 0
\tag{9}
$$

$$
\frac{d^2\bar{T}}{dy^2} - \Pr s\, \bar{T} = 0
\tag{10}
$$

Therefore, the solutions of (9) and (10) in Laplace domain are:

$$
\bar{u} = \frac{1}{s} exp\left[-y\sqrt{(m_0 + s)}\right] + \frac{G}{(s\,\Pr - m_0 - s)} \frac{\left[exp(-y\sqrt{sPr}) - exp[-y\sqrt{(m_0+s)}]\right]}{s\sqrt{s}}
\tag{11}
$$

$$
\bar{T} = \frac{1}{s\sqrt{sPr}\, exp(-y\sqrt{sPr})}
\tag{12}
$$

To obtain the velocity and temperature distributions in the time domain, (11) and (12) are to be inverted, since these equations are difficult to be inverted in closed form, we use a numerical procedure used by [17] which based on the Riemann-sum approximation (RSA) method.
Laplace domain can be inverted to the time domain t as follows:

$$
u(y,t) = \frac{e^{\epsilon t}}{t}\left[\frac{1}{2}\bar{u}(y,\epsilon) + Re\sum_{k=1}^N \bar{u}\left(y, \epsilon + \frac{ik\pi}{t}\right)(-1)^k\right]
\tag{13}
$$

$$
\theta(y,t) = \frac{e^{\epsilon t}}{t}\left[\frac{1}{2}\bar{\theta}(y,\epsilon) + Re\sum_{k=1}^N \bar{\theta}\left(y, \epsilon + \frac{iK\pi}{t}\right)(-1)^k\right]
\tag{14}
$$

where N is the number of terms used in the Riemann-sum approximation (RSA), $\in$ is the real part of the Bromwich contour that is used in inverting Laplace transforms and Re refers to the real part of is the imaginary number i. The RSA for the Laplace inversion involves a single summation for the numerical process. The correctness of this method depends on the value of $\in$ and its truncation error dictated by N. Based on numerical experimentation and according to Tzou [18], the value of $\in$ must be selected so that the Bromwich contour encompasses all the branch points. For quicker convergence, the quantity $\in t = 4.7$ gives the most satisfactory results.

"Skin-friction"

The skin friction $\left(\tau\right)$ are gotten as:

$$\tau = \frac{du}{dy}\bigg|_{y=0} = -\frac{\sqrt{m_0+s}}{s} + \frac{G\left(\sqrt{sPr}-\sqrt{m_0+s}\right)}{s\sqrt{sPr}\left(sPr-m_0-s\right)} \tag{15}$$

The solutions obtained are in Laplace domain and are converted to time domain by applying the RSA stated as shown in (13) and (14).

3.　RESULTS AND DISCUSSION

For validation of accuracy of RSA method, we employed MATLAB software to obtain numerical values of the dimensionless velocity and skin friction and compared our results with those presented by Sacheti *et al.* [2]. Figure 1 exhibits the role of non-dimensional time (t) and Prandtl number (Pr) on velocity profile. It is clear from Figure 1 that, increasing t, leads to a corresponding increase in the fluid velocity for both Prandtl numbers $(Pr = 0.71)$ and $(Pr = 7.0)$ but, it is more prominent for the case of air. This is because increase in temperature increases the convection current so that velocity also increases with time.
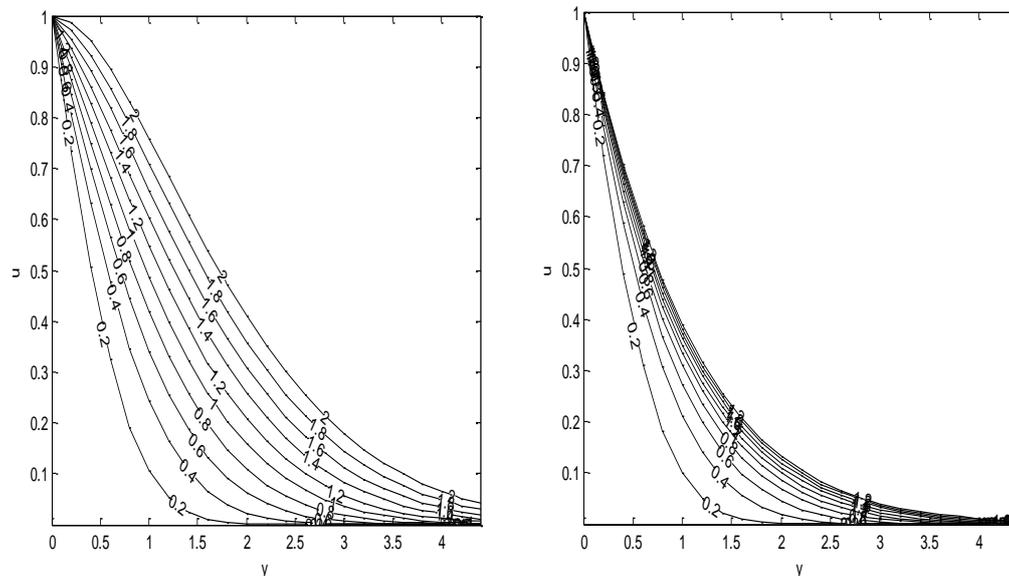
Figure 1: "Velocity profile showing the effect of t for Pr =0.71 and Pr =7.0" respectively, with fixed values of $(Gr = 1.0 \, m_0 = 0.5)$

Figure 2 illustrates the impact of Hartmann number (m_0) and Pr on velocity profile. It is detected that for both Prandtl numbers $(Pr = 0.71)$ and $(Pr = 7.0)$ that, increase in magnetic field has a tendency to retard the motion of fluid flow in the vertical plate. This is because the application of magnetic field creates a resistive force similar to the drag force that acts in the opposite direction of the fluid motion, thus causing the fluid velocity to decreases. Figure 3 presents the role of buoyancy force on fluid velocity for different values of $(Pr = 0.71, 7.0)$. It is realized that, fluid velocity increases as buoyancy force increases for both Prandtl numbers, but the significance effect is more noticeable in the case of air compare to water.
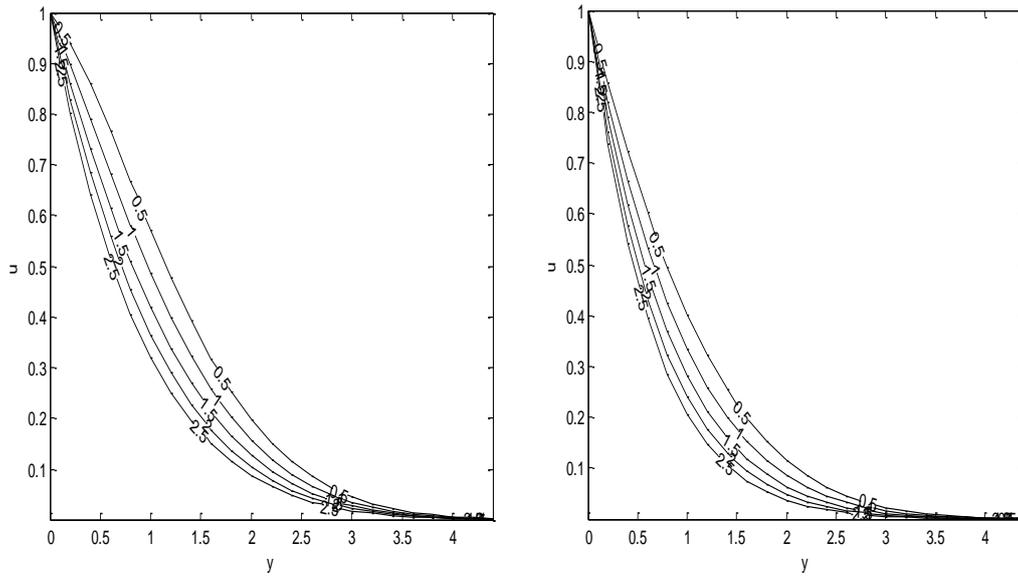
Figure 2. Velocity profile showing the effect of m_0 for Pr =0.71 and Pr =7.0" respectively, with fixed values of ($t = 1.0 Gr = 1.0$)
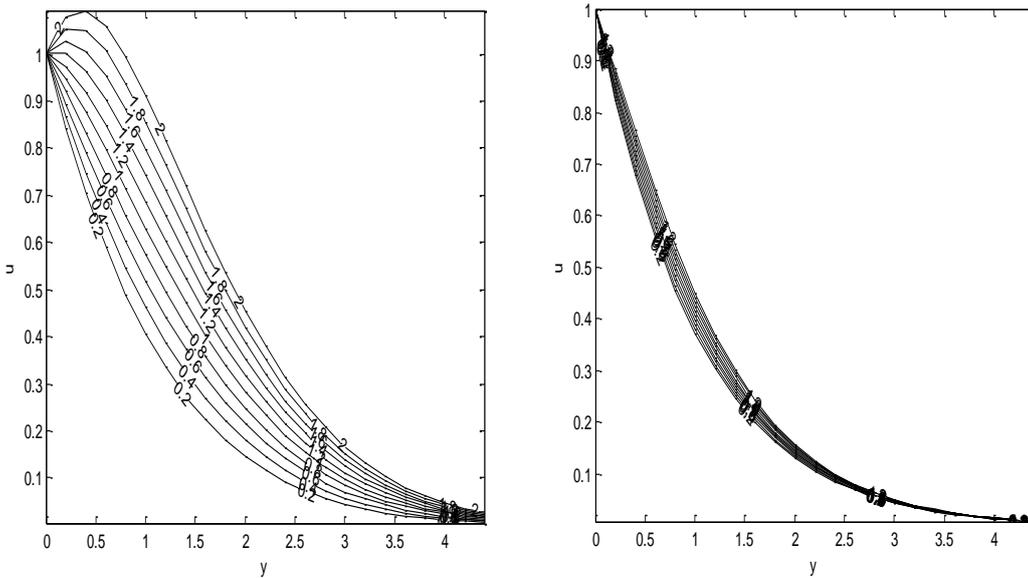
Figure 3. Velocity profile showing the effect of Gr for Pr =0.71 and Pr =7.0 respectively, with fixed values of ($t = 1.0, m_0 = 0.5$)

Figure 4 displays the effect of magnetic field parameter and t on the skin-friction. It is observed that, skin friction declines as the Hartmann number rises for both Prandtl numbers. This is physically true since increasing magnetic field strength decreases fluid velocity which in turn decreases skin-friction at the wall. Similar results have been found by [19-24]. Although this has been wrongly presented in Table 2 of Sacheti *et al.* [2] that skin-friction increases with increase in Hartmann number. The above correction will help design engineers and serve as improvement on unsteady flow formation in a vertical channel with transversely applied magnetic field. Figure 5 depicts the role of buoyancy parameter and time on skin friction. It is found that, skin friction increases as buoyancy force increases regardless of the working fluid. Finally, a comparison was made between the present work and the work of [2], it is detected that, the numerical results obtained for the velocity is found to be in excellent agreement as shown in Table 1 while there is discrepancy in the skin friction which has been addressed in Table 2.
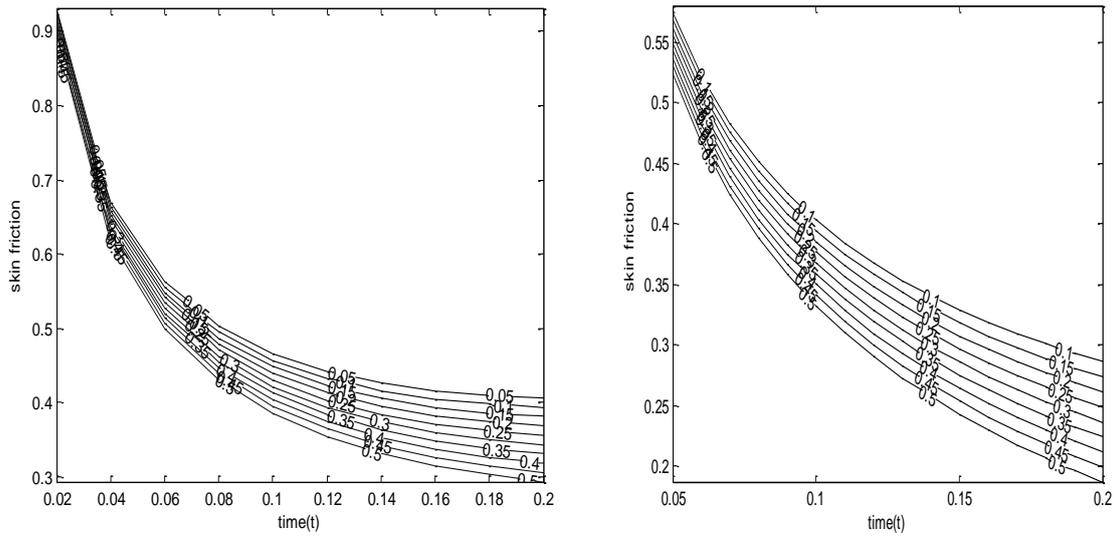
Figure 4. Variation of skin friction for different values of m_0 when Pr =0.71 and Pr =7.0" respectively, with fixed value of $Gr = 1.0$
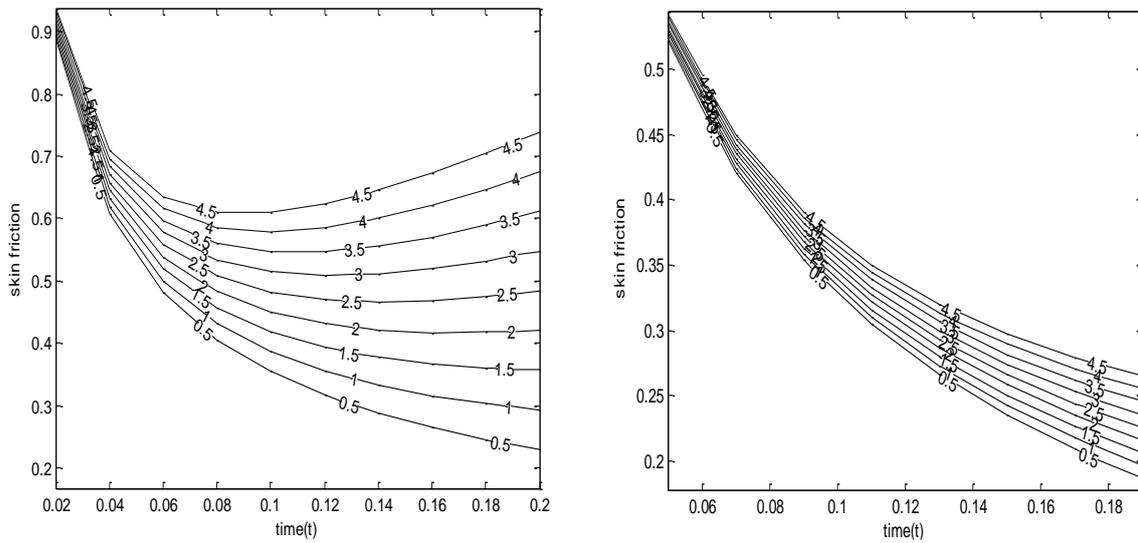
Figure 5. Variation of skin friction for different values of Gr when Pr =0.71 and Pr =7.0 respectively, with fixed value of $m_0 = 0.5$

Table 1. Comparison of the numerical values of the velocity obtained using the Riemann-sum approximation method and that obtained using the convolution integration method by [2]

P	G	y	m_0	Sacheti *et al.* [2]	Present work	P	G	y	m_0	Sacheti *et al.* [2]	Present work
2.0	1.0	0.5	1.0	0.7761	0.7761	7.0	1.0	0.5	1.0	0.7236	0.7236
			2.0	0.7586	0.7585				2.0	0.7071	0.7071
			3.0	0.7419	0.7419				3.0	0.6913	0.6912
		1.0	1.0	0.5202	0.5202			1.0	1.0	0.4703	0.4703
			2.0	0.5002	0.5003				2.0	0.4517	0.4517
			3.0	0.4818	0.4818				3.0	0.4341	0.4341
	3.0	0.5	1.0	0.9149	0.9149		3.0	0.5	1.0	0.7576	0.7576
			2.0	0.8951	0.8951				2.0	0.7405	0.7405
			3.0	0.8762	0.8762				3.0	0.7242	0.7242
		1.0	1.0	0.6402	0.6402			1.0	1.0	0.4906	0.4905
			2.0	0.6178	0.6177				2.0	0.4716	0.4716
			3.0	0.5965	0.5965				3.0	0.4535	0.4535

Table 2. Comparison of the numerical values of the skin friction (τ) obtained using the Riemann-sum approximation method and that obtained using the convolution integrals method by [2].

Gr	m_0	t	Sacheti *et al.* [2]	Present work	Gr	m_0	t	Sacheti *et al.* [2]	Present work
1.0	0.0	0.1	1.8256	0.4215	2.0	0.0	0.1	1.8671	0.4318
		0.2	1.3445	0.3114			0.2	1.4274	0.3222
		0.3	1.1545	0.2685			0.3	1.2789	0.2996
		0.4	1.0579	0.2470			0.4	1.2238	0.2885
	1.0	0.1	1.9704	0.2459		1.0	0.1	1.9811	0.2562
		0.2	1.5281	0.0668			0.2	1.5505	0.0873
		0.3	1.3593	-0.2958			0.3	1.3941	0.0038
		0.4	1.5786	-0.0890			0.4	1.3230	-0.0486
	2.0	0.1	2.1407	0.0758		2.0	0.1	2.1519	0.0861
		0.2	1.7591	0.1628			0.2	1.7830	-0.1426
		0.3	1.6315	-0.2958			0.3	1.6700	-0.2659
		0.4	1.5786	-0.3870			0.4	1.633	-0.3476
	3.0	0.1	2.3057	-0.0889		3.0	0.1	2.3173	-0.2385
		0.2	1.9767	-0.3790			0.2	2.0042	-0.5637
		0.3	1.8819	-0.5430			0.3	1.9242	-0.7427
		0.4	1.8520	-0.6548			0.4	1.9135	-0.8602

4. CONCLUSION

In this present work, a scrutiny of the impulsive motion of an electrically conducting fluid near an infinite vertical plate has been done. The governing equations have been solved by combining the LT technique to transform the PDE to ODE, while the RSA approach is employed to transform the closed-form solutions from Laplace domain to time domain. From this investigation, we observed that, the velocity obtained by the current technique are in precise agreement with those described by Sacheti *et al* in Table 1 while there is discrepancy in the skin friction with the result of published work of Sacheti *et al.* which has been addressed in Table 2. Results show that less computational time is incurred by using the Riemann-sum approximation approach.

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